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Summations for Basic Hypergeometric Series Involving a q-Analogue of the Digamma Function

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Abstract—Using a simple method, numerous summation formulas for hypergeometric and basic hypergeometric series are derived. Among these summation formulas are nonterminating extensions and q-extensions of identities recorded by Lavoie, Luke, Watson, and Srivastava. At the result side of the basic hypergeometric summations, there appears a q-analogue of the digamma function. Some of its properties are also studied.

Keywords—Basic hypergeometric series, q-digamma function, q-polygamma function, Summation formulas.

1. INTRODUCTION AND DESCRIPTION OF THE METHOD

In [1–4] a simple identity for hypergeometric series is combined with a limiting process to obtain summation formulas for hypergeometric series in which the digamma function appears at the result side. In this paper, we start with the q-analogue of this identity. Again by limiting processes, we derive numerous identities for basic hypergeometric series, among which are nonterminating extensions and q-analogues of the identities in [1–4]. Not surprisingly, at the result side of our identities there appears a q-analogue of the digamma function. Letting q tend to 1, also several new summations for hypergeometric series are obtained.

We adopt the usual conventions of writing basic hypergeometric series (cf. [5, Section 1.2]), $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$, $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$,

$$ {}_r\Phi_s\left[\begin{matrix} a_1,\dots,a_r; \\ b_1,\dots,b_s; \end{matrix} q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^n, $$

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX and the short notation (cf. [5, Section 2.1])

$$ {}_{r+1}W_r(a;a_1,\dots,a_{r-2};q,z) := {}_{r+1}\Phi_r\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a_1,\dots,a_{r-2}; \\ \sqrt{a}, -\sqrt{a}, aq/a_1,\dots,aq/a_{r-2}; \end{matrix} q, z\right] $$

for the very well-poised basic hypergeometric series.

The hypergeometric notations (cf. [6, Section 2.1; 7, Section 2.1]) are $(a)_n = \prod_{i=0}^{n-1}(a+i)$,

$$ {}_rF_s\left[\begin{matrix} a_1,\dots,a_r; \\ b_1,\dots,b_s; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_r)_n}{n!\,(b_1)_n\cdots(b_s)_n} z^n , $$

and the short notation

$$ V(a;a_1,\dots,a_{r-1};z) := {}_{r+1}F_r\left[\begin{matrix} a, 1+a/2, a_1,\dots,a_{r-1}; \\ a/2, 1+a-a_1,\dots,1+a-a_{r-2}; \end{matrix} z\right] $$

for the very well-poised hypergeometric series.

All identities in our paper are subject to suitable conditions on the parameters such that the involved hypergeometric or basic hypergeometric series converge. We shall not state these conditions for each identity. The reader should consult [5, pp. 4–5].

We shall frequently use the short notations

$$ (a_1,\dots,a_m;q)_\beta := \prod_{i=1}^{m}(a_i;q)_\beta, $$
$$ (a_1,\dots,a_m)_\beta := \prod_{i=1}^{m}(a_i)_\beta, $$

and

$$ \Gamma\left[\begin{matrix} a_1,\dots,a_m \\ b_1,\dots,b_n \end{matrix}\right] := \frac{\prod_{i=1}^{m}\Gamma(a_i)}{\prod_{i=1}^{n}\Gamma(b_i)}, $$

where $\Gamma(z)$ is the gamma function.

Now let us describe our method.

The aforementioned q-analogue of the identity utilized in [1–4] is

$$ {}_{r+1}\Phi_{s+1}\left[\begin{matrix} qa_1,\dots,qa_r, q; \\ qb_1,\dots,qb_s, q^2; \end{matrix} q, z\right] = \frac{(1-b_1)\cdots(1-b_s)(1-q)}{z\,(1-a_1)\cdots(1-a_r)}\left({}_r\Phi_s\left[\begin{matrix} a_1,\dots,a_r; \\ b_1,\dots,b_s; \end{matrix} q, z\right] - 1\right). \tag{1.1} $$

It is straightforward to verify this identity. Suppose that the parameters in this identity are chosen such that the resulting ${}_r\Phi_s$-series at the right-hand side of (1.1) can be summed. By letting a_1, say, approach to 1 and by computing the limit at the right-hand side by application of l'Hôspital's rule, we then obtain a summation formula for the resulting basic hypergeometric series at the left-hand side.

Because of the application of l'Hôspital's rule, we have to deal with derivations of infinite expressions $(a;q)_\infty$. This results in the appearance of a q-analogue of the digamma function. Though this q-analogue previously appeared in the literature [8–10], it does not seem to have been studied so far. Therefore, as a preparation, in the next section we derive some of its properties. In the subsequent sections we list the identities that are obtained by our method: in Section 3, those that are derived by summing the ${}_r\Phi_s$ at the right-hand side of (1.1) with the help of the nonterminating q-Pfaff-Saalschütz sum [5, equation (2.10.12)], in Section 4 those that are derived by using Bailey's nonterminating extension of Jackson's ${}_8\Phi_7$-sum [5, equation (2.11.7)], in Section 5 those that are derived by using Verma and Jain's [11, equation (1.6)] q-analogue of Whipple's ${}_3F_2$-sum, and in Section 6 those that are derived by using Verma and Jain's [11, equation (1.5)] q-analogue of Watson's ${}_3F_2$-sum.

2. THE q-DIGAMMA FUNCTION

Thomae [12] and Jackson [13] introduced the following q-analogue of the gamma function (cf. [5, Section 1.10, Example 1.23]):

$$\Gamma_q(x) = \begin{cases} \dfrac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} & |q|<1 \\ \dfrac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}} & 1<q. \end{cases} \tag{2.1}$$

For properties of the q-gamma function, see [5,8–10,12–17]. It was shown by Askey [8] and Moak [10] that for $x \in \mathbb{R}$ (set of real numbers), $x \neq 0,-1,-2,\dots$, the q-gamma function $\Gamma_q(x)$ indeed tends to the gamma function $\Gamma(x)$ when letting $q \to 1$.

The digamma function $\psi(x)$ (see, e.g., [18, Section 1.7]) by definition is the logarithmic derivative of $\Gamma(x)$,

$$\psi(x) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}.$$

In view of this, it is natural to define a q-digamma function $\psi_q(x)$ by

$$\psi_q(x) := \frac{\frac{d}{dx}\Gamma_q(x)}{\Gamma_q(x)}, \tag{2.2}$$

or more explicitly,

$$\psi_q(x) := \begin{cases} -\log(1-q) + \log q \displaystyle\sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}} & |q|<1 \\ -\log(q-1) + \left(x-\dfrac{1}{2}\right)\log q + \log q \displaystyle\sum_{n=0}^{\infty} \frac{q^{-n-x}}{q^{-n-x}-1} & 1<q. \end{cases} \tag{2.3}$$

However, before being allowed to call $\psi_q(x)$ a q-analogue of the digamma function, we have to show that $\psi_q(x)$ tends to $\psi(x)$ when letting $q \to 1$. In Theorem 1 below, we prove this fact for real q, which suffices for our purposes. It should not be too difficult to prove the same for complex q, $|q| \neq 1$.

THEOREM 1. *Let $q \in \mathbb{R}$. For $x \in \mathbb{C}$ (set of complex numbers), $x \neq 0,-1,-2,\dots$, the following holds:*

$$\lim_{q\to 1} \psi_q(x) = \psi(x).$$

PROOF. First let $0<q<1$. Write $\psi_q(x)$ in the form

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{n+1}}{1-q^{n+1}} - \log q \sum_{n=0}^{\infty} \frac{q^{n+1}\left(1-q^{x-1}\right)}{\left(1-q^{n+1}\right)\left(1-q^{n+x}\right)}. \tag{2.4}$$

We shall show that

$$\lim_{q\uparrow 1}\left(\log(1-q) - \log q \sum_{n=0}^{\infty} \frac{q^{n+1}}{1-q^{n+1}}\right) = \gamma, \tag{2.5}$$

where γ is the Euler gamma (cf. [18, 1.1.(4); 19, Section 12.1]), and

$$\lim_{q\uparrow 1}\left(-\log q \sum_{n=0}^{\infty} \frac{q^{n+1}\left(1-q^{x-1}\right)}{\left(1-q^{n+1}\right)\left(1-q^{n+x}\right)}\right) = \sum_{n=0}^{\infty} \frac{x-1}{(n+1)(n+x)}. \tag{2.6}$$

Supposing that this would be done, we would have

$$\lim_{q\uparrow 1} \psi_q(x) = -\gamma + \sum_{n=0}^{\infty} \frac{x-1}{(n+1)(n+x)}.$$

The expression at the right-hand side of this equation is just one of the many ways of writing the digamma function $\psi(x)$ (cf. [18, 1.7.(3)]). This would prove $\lim_{q\uparrow 1} \psi_q(x) = \psi(x)$.

We start with proving (2.5). In the first step we rewrite the expression at the left-hand side of (2.5). We have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(-\frac{q^n \log q}{1-q^n} - \log\frac{1-q^{n+1}}{1-q^n}\right) &= -\log q \sum_{n=0}^{\infty}\frac{q^{n+1}}{1-q^{n+1}} - \sum_{n=1}^{\infty}\left(\log\left(1-q^{n+1}\right) - \log\left(1-q^n\right)\right)\\ &= -\log q \sum_{n=0}^{\infty}\frac{q^{n+1}}{1-q^{n+1}} + \log(1-q).\end{aligned}$$

Thus, (2.5) is equivalent with

$$\lim_{q\uparrow 1}\sum_{n=1}^{\infty}\left(-\frac{q^n \log q}{1-q^n} - \log\frac{1-q^{n+1}}{1-q^n}\right) = \gamma. \tag{2.7}$$

Suppose that we are allowed to interchange limit and summation at the left-hand side of (2.7). This would give

$$\lim_{q\uparrow 1}\sum_{n=1}^{\infty}\left(-\frac{q^n \log q}{1-q^n} - \log\frac{1-q^{n+1}}{1-q^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \log\frac{n+1}{n}\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right).$$

But the right-hand side of this equation by definition equals γ (cf. [18, 1.1.(4); 19, p. 235]); hence, equation (2.7) would be established, and hence also (2.5).

In order to justify this interchange of limit and sum, we claim that for $0 < q < 1$ we have

$$0 \le -\frac{q^n \log q}{1-q^n} - \log\frac{1-q^{n+1}}{1-q^n} \le \frac{1}{n} - \log\frac{n+1}{n}. \tag{2.8}$$

Because of $1/n - \log((n+1)/n) \le 1/n^2$ (see [19, p. 235]), this would imply that the sum at the left-hand side of (2.7) is absolutely convergent, uniformly in q. Therefore, we are allowed to interchange limit and sum.

For the proof of (2.8), let us abbreviate

$$h(q) := -\frac{q^n \log q}{1-q^n} - \log\frac{1-q^{n+1}}{1-q^n}.$$

Clearly we have $\lim_{q\downarrow 0} h(q) = 0$ and $\lim_{q\uparrow 1} h(q) = 1/n - \log((n+1)/n)$. Thus, (2.8) would be established if we can show that $h(q)$ is an increasing function for $0 < q < 1$.

Let us compute the derivative of $h(q)$,

$$\frac{d}{dq}h(q) = -\frac{n\,q^{n-1}}{(1-q^n)^2}\left(\frac{n+1}{n}\frac{(1-q)\,(1-q^n)}{(1-q^{n+1})} + \log q\right).$$

In order to show that $h(q)$ is increasing we should have $\frac{d}{dq}\,h(q) \ge 0$; i.e.,

$$\frac{n+1}{n}\frac{(1-q)\,(1-q^n)}{(1-q^{n+1})} + \log q \le 0, \qquad 0 < q < 1. \tag{2.9}$$

Again, let us abbreviate

$$g(q) := \frac{n+1}{n}\frac{(1-q)\,(1-q^n)}{(1-q^{n+1})} + \log q.$$

The derivation of $g(q)$ is

$$\frac{d}{dq}g(q) = \frac{(1-q)^2 q^n}{n\,q\,(1-q^{n+1})^2}\left(n - \left(q+q^2+\cdots+q^n\right)\right)\left(\left(q^{-1}+q^{-2}+\cdots+q^{-n}\right)-n\right),$$

which is greater than 0 for $0<q<1$. Together with $\lim_{q\uparrow 1} g(q) = 0$, this implies that $g(q) < 0$ for $0 < q < 1$. That is, the inequality (2.9) holds, and hence also (2.8). This finishes the proof of (2.5).

For the proof of (2.6), we proceed as follows:

$$\begin{aligned}\lim_{q\uparrow 1}\left(-\log q\sum_{n=0}^{\infty}\frac{q^{n+1}(1-q^{x-1})}{(1-q^{n+1})\,(1-q^{n+x})}\right) &= \lim_{q\uparrow 1}\left(-\frac{\log q}{1-q}\right)\lim_{q\uparrow 1}\sum_{n=0}^{\infty} q^{n+1}\frac{(1-q)\,(1-q^{x-1})}{(1-q^{n+1})\,(1-q^{n+x})}\\ &= 1\cdot\sum_{n=0}^{\infty}\frac{x-1}{(n+1)(n+x)}.\end{aligned}$$

The interchange of limit and summation can be justified in a similar manner as above in the proof of (2.5). We omit the details.

Finally we show $\lim_{q\downarrow 1}\psi_q(x) = \psi(x)$. If $q > 1$, we can rewrite (2.3) by

$$\psi_q(x) = \left(x-\frac{3}{2}\right)\log q - \log\left(1-\frac{1}{q}\right) + \log\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^{-n-x}}{1-q^{-n-x}} = \left(x-\frac{3}{2}\right)\log q + \psi_{1/q}(x).$$

From this and the fact that we have already shown $\lim_{q\uparrow 1}\psi_q(x) = \psi(x)$, we get

$$\lim_{q\downarrow 1}\psi_q(x) = \lim_{q\downarrow 1}\left(\left(x-\frac{3}{2}\right)\log q\right) + \lim_{q\uparrow 1}\psi_q(x) = 0 + \psi(x) = \psi(x).$$

This completes the proof of Theorem 1. ∎

From now on we always assume q to be a complex number with $|q| < 1$.

For our purposes, because of notational convenience, it will be useful to consider a "logarithmic-free" variant of $\psi_q(x)$, namely

$$\tilde{\psi}_q(x) := -(1-q)\sum_{n=0}^{\infty}\frac{xq^n}{1-xq^n}. \tag{2.10}$$

This definition makes sense for complex q, x, $|q| < 1$, and $x \neq 1, q^{-1}, q^{-2}, \ldots$. $\tilde{\psi}_q(x)$ is related to $\psi(x)$ by

$$\tilde{\psi}_q(q^x) = -\frac{1-q}{\log q}\psi_q(x) - (1-q)\frac{\log(1-q)}{\log q}. \tag{2.11}$$

So the limit $\lim_{q\uparrow 1}\tilde{\psi}_q(q^x)$ does not even exist, but at least, because of (2.11) and Theorem 1, we have

$$\lim_{q\uparrow 1}\left(\tilde{\psi}_q\left(q^x\right) - \tilde{\psi}_q\left(q^y\right)\right) = \psi(x) - \psi(y), \tag{2.12}$$

which suffices for our purposes.

The next theorem lists miscellaneous properties of the q-digamma function $\tilde{\psi}_q(x)$. These will be of relevance in the subsequent sections when manipulating q-digamma functions and when taking limits of q-digamma functions.

THEOREM 2. *Let N denote a nonnegative integer. The following identities hold:*

$$\tilde{\psi}_q(x) - \tilde{\psi}_q(y) = \sum_{n=0}^{\infty} \frac{q^n(y-x)(1-q)}{(1-xq^n)(1-yq^n)} \tag{2.13a}$$

$$= \frac{(1-q)(y-x)}{(1-x)(1-y)} {}_3\Phi_2\left[\begin{matrix} x, y, q; \\ qx, qy; \end{matrix} q, q\right], \tag{2.13b}$$

$$\tilde{\psi}_q(x) - \tilde{\psi}_q\left(x q^N\right) = -(1-q) \sum_{n=0}^{N-1} \frac{xq^n}{1-xq^n}, \tag{2.14}$$

$$\tilde{\psi}_{q^2}(x) + \tilde{\psi}_{q^2}(qx) = (1+q)\tilde{\psi}_q(x), \tag{2.15}$$

$$\tilde{\psi}_{q^2}\left(x^2\right) = \frac{(1+q)}{2}\left(\tilde{\psi}_q(x) + \tilde{\psi}_q(-x)\right), \tag{2.16}$$

$$\lim_{q\uparrow 1}\left(\tilde{\psi}_q\left(q^x\right) - \tilde{\psi}_q\left(q^y\right)\right) = \psi(x) - \psi(y), \tag{2.17}$$

$$\lim_{q\uparrow 1}\left(\tilde{\psi}_q(-q^x) - \tilde{\psi}_q(-q^y)\right) = 0, \tag{2.18}$$

$$\lim_{N\to\infty}\left(\tilde{\psi}_q\left(\frac{x}{q^N}\right) - \tilde{\psi}_q\left(\frac{y}{q^N}\right)\right) = \tilde{\psi}_q(x) - \tilde{\psi}_q(y) + \tilde{\psi}_q\left(\frac{q}{y}\right) - \tilde{\psi}_q\left(\frac{q}{x}\right). \tag{2.19}$$

PROOF. The identities (2.13)–(2.16) follow directly from the definition (2.10). Identity (2.17) is a restatement of (2.12).

In order to prove (2.18), we use (2.13a) to get

$$\tilde{\psi}_q\left(-q^x\right) - \tilde{\psi}_q\left(-q^y\right) = \sum_{n=0}^{\infty} \frac{q^n\left(q^x - q^y\right)(1-q)}{\left(1+q^{n+x}\right)\left(1+q^{n+y}\right)}.$$

We may choose q_0, $0 < q_0 < 1$, such that $|(1+q^{n+x})^{-1}(1+q^{n+y})^{-1}| \le 1$ for all n and $q_0 < q < 1$. This implies

$$\left|\sum_{n=0}^{\infty} \frac{q^n\left(q^x - q^y\right)(1-q)}{\left(1+q^{n+x}\right)\left(1+q^{n+y}\right)}\right| \le \sum_{n=0}^{\infty} q^n\left|q^x - q^y\right|(1-q) \le \left|q^x - q^y\right|, \tag{2.20}$$

provided $q_0 < q < 1$. For $q \uparrow 1$, the right-hand side in (2.20) tends to 0, and hence so does the left-hand side. Thus, (2.14) is established.

For the proof of (2.19) we note that by (2.14) we may write

$$\tilde{\psi}_q\left(\frac{x}{q^N}\right) - \tilde{\psi}_q(x) - \left(\tilde{\psi}_q\left(\frac{y}{q^N}\right) - \tilde{\psi}_q(y)\right) = \sum_{n=0}^{N-1} \frac{q^n(1-q)(q/x - q/y)}{\left(1-q^{n+1}/x\right)\left(1-q^{n+1}/y\right)}.$$

Therefore, by appealing to (2.13a), we obtain

$$\lim_{N\to\infty}\left(\tilde{\psi}_q\left(\frac{x}{q^N}\right) - \tilde{\psi}_q(x) - \tilde{\psi}_q\left(\frac{y}{q^N}\right) + \tilde{\psi}_q(y)\right) = \tilde{\psi}_q\left(\frac{q}{y}\right) - \tilde{\psi}_q\left(\frac{q}{x}\right),$$

which is equivalent with (2.19). ∎

In our identities, the following q-analogue of the first polygamma function (cf. [18, 1.16]) will also appear:

$$\tilde{\psi}_q^{(1)}(x) := (1-q)^2 \sum_{n=0}^{\infty} \frac{xq^n}{(1-xq^n)^2}. \tag{2.21}$$

The following theorem lists some relevant properties of $\tilde{\psi}_q^{(1)}(x)$.

THEOREM 3. *The following identities hold:*

$$\lim_{q\uparrow 1} \tilde{\psi}_q^{(1)}\left(q^x\right) = \psi^{(1)}(x), \tag{2.22}$$

where $\psi^{(1)}(x)$ denotes the first polygamma function,

$$\lim_{q\uparrow 1} \tilde{\psi}_q^{(1)}\left(-q^x\right) = 0, \tag{2.23}$$

$$\lim_{N\to\infty} \tilde{\psi}_q^{(1)}\left(xq^{-N}\right) = \tilde{\psi}_q^{(1)}(x) + \tilde{\psi}_q^{(1)}\left(\frac{q}{x}\right), \tag{2.24}$$

$$\tilde{\psi}_{q^2}^{(1)}(x) + \tilde{\psi}_{q^2}^{(1)}(qx) = (1+q)^2 \tilde{\psi}_q^{(1)}(x). \tag{2.25}$$

SKETCH OF PROOF. We have

$$\lim_{q\uparrow 1} \tilde{\psi}_q^{(1)}\left(q^x\right) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2}.$$

The right-hand side in this equation equals $\psi^{(1)}(x)$ (cf. [18, 1.16.(9)]). The proof of (2.23) is analogous to that of (2.18) and is therefore omitted. Identity (2.25) follows directly from definition (2.21).

Identity (2.24) is established by the following sequence of manipulations:

$$\begin{aligned}
\lim_{N\to\infty} \tilde{\psi}_q^{(1)}\left(xq^{-N}\right) &= \lim_{N\to\infty} (1-q)^2 \left(\sum_{n=0}^{N-1} \frac{xq^{-N+n}}{\left(1-xq^{-N+n}\right)^2} + \sum_{n=0}^{\infty} \frac{xq^n}{\left(1-xq^n\right)^2}\right) \\
&= \lim_{N\to\infty} (1-q)^2 \sum_{n=1}^{N} \frac{xq^{-n}}{\left(1-xq^{-n}\right)^2} + \tilde{\psi}_q^{(1)}(x) \\
&= \lim_{N\to\infty} (1-q)^2 \sum_{n=1}^{N} \frac{q^n/x}{\left(1-q^n/x\right)^2} \\
&= \tilde{\psi}_q^{(1)}(x) + \tilde{\psi}_q^{(1)}\left(\frac{q}{x}\right).
\end{aligned}$$

∎

3. IDENTITIES RELATED TO THE NONTERMINATING q-PFAFF-SAALSCHÜTZ SUM

In (1.1) we choose $r=3$, $s=2$, $a_1=a$, $a_2=b$, $a_3=c$, $b_1=e$, $b_2=qabc/e$, $z=q$, and apply the nonterminating extension of the q-Pfaff-Saalschütz sum [5, equation (II.24)] to the resulting ${}_3\Phi_2$ at the right-hand side. This gives

$$\begin{aligned}
{}_4\Phi_3\left[\begin{matrix} aq, bq, cq, q; \\ eq, abcq^2/e, q^2; \end{matrix} q, q\right] &= \frac{1}{q} \frac{(1-e)(1-qabc/e)(1-q)}{(1-a)(1-b)(1-c)} \left(\frac{(q/e, bcq/e, acq/e, abq/e; q)_\infty}{(aq/e, bq/e, cq/e, abcq/e; q)_\infty} - 1\right. \\
&\quad \left. - \frac{\left(q/e, a, b, c, abcq^2/e^2; q\right)_\infty}{(e/q, aq/e, bq/e, cq/e, abcq/e; q)_\infty} {}_3\Phi_2\left[\begin{matrix} aq/e, bq/e, cq/e; \\ q^2/e, abcq^2/e^2; \end{matrix} q, q\right]\right).
\end{aligned}$$

Now let $c \to 1$. The limit $\lim_{c\to 1}(1/(1-c))((\dots)_\infty/(\dots)_\infty - 1)$, which has to be computed at the right-hand side, is obtained by using l'Hôspital's rule, while the limit of the other expressions can be obtained directly. Thus, we obtain

$$\begin{aligned}
{}_4\Phi_3\left[\begin{matrix} aq, cq, q, q; \\ bq, acq^2/b, q^2; \end{matrix} q, q\right] &= \frac{1}{q} \frac{(1-b)(1-qac/b)}{(1-a)(1-c)} \left(\tilde{\psi}_q\left(\frac{q}{b}\right) - \tilde{\psi}_q\left(\frac{cq}{b}\right) - \tilde{\psi}_q\left(\frac{aq}{b}\right) + \tilde{\psi}_q\left(\frac{acq}{b}\right)\right. \\
&\quad \left. - \frac{(1-q)\left(a, c, q, acq^2/b^2; q\right)_\infty}{(b/q, aq/b, cq/b, acq/b; q)_\infty} {}_3\Phi_2\left[\begin{matrix} aq/b, cq/b, q/b; \\ q^2/b, acq^2/b^2; \end{matrix} q, q\right]\right),
\end{aligned} \tag{3.1}$$

where we replaced b by c and then e by b. Replacing a by q^a, b by q^b, c by q^c and then letting $q \uparrow 1$, with the help of (2.17) we get the following nonterminating extension of [4, equations (3.7)–(3.10)]:

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a+1, c+1, 1, 1; \\ b+1, 2+a+c-b, 2; \end{matrix} 1\right] \\
&= \frac{b(1+a+c-b)}{ac}\Big(\psi(1-b) - \psi(1+c-b) - \psi(1+a-b) + \psi(1+a+c-b) \\
&\quad - \Gamma\left[\begin{matrix} b-1, 1+a-b, 1+c-b, 1+a+c-b \\ a, c, 2+a+c-2b \end{matrix}\right] {}_3F_2\left[\begin{matrix} 1+a-b, 1+c-b, 1-b; \\ 2-b, 2+a+c-2b; \end{matrix} 1\right]\Big).
\end{aligned} \tag{3.2}
$$

Setting $c = 0$ in (3.1) gives a q-analogue for [4, equation (3.11)],

$$
{}_3\Phi_2\left[\begin{matrix} aq, q, q; \\ bq, q^2; \end{matrix} q, q\right] = \frac{1}{q}\frac{(1-b)}{(1-a)}\left(\tilde{\psi}_q\left(\frac{q}{b}\right) - \tilde{\psi}_q\left(\frac{aq}{b}\right) - \frac{(1-q)\,(a,q;q)_\infty}{(b/q, aq/b; q)_\infty} {}_2\Phi_1\left[\begin{matrix} aq/b, q/b; \\ q^2/b; \end{matrix} q, q\right]\right). \tag{3.3}
$$

In order to obtain a nicer q-analogue of [4, equation (3.11)], in (3.1) set $c = q^{-N}$ and then let $N \to \infty$. By (2.19) this yields

$$
{}_3\Phi_2\left[\begin{matrix} aq, q, q; \\ bq, q^2; \end{matrix} q, \frac{b}{a}\right] = \frac{(1-1/b)}{(1-1/a)}\left(\tilde{\psi}_q(b) - \tilde{\psi}_q\left(\frac{b}{a}\right)\right). \tag{3.4}
$$

Writing this in the form

$$
\tilde{\psi}_q(b) - \tilde{\psi}_q\left(\frac{b}{a}\right) = \frac{(a)_\infty}{(b)_\infty}\sum_{n=1}^{\infty}\frac{(1-q)}{(1-q^n)}\frac{(bq^n)_\infty}{(aq^n)_\infty}\left(\frac{b}{a}\right)^n, \tag{3.5}
$$

we also have a perfect q-analogue for [4, equation (3.12)].

It might be observed that (3.3) could also be obtained by combining a special case of a three-term relation for ${}_3\Phi_2$-series [5, Appendix, (III.34), $a \to q$, $b \to q/b$, $c \to aq/b$, $d \to q^2/b$, $e \to aq^2/b$] with (2.13b), while (3.4) could be obtained by combining the ${}_3\Phi_2$-transformation [5, Appendix, (III.9), $a \to q$, $b \to aq$, $c \to q$, $d \to bq$, $e \to q^2$] with (2.13b).

4. IDENTITIES RELATED TO BAILEY'S NONTERMINATING JACKSON SUM

In (1.1) choose $r = 8$, $s = 7$, $a_1 = a$, $a_2 = q\sqrt{a}$, $a_3 = -q\sqrt{a}$, $a_4 = b$, $a_5 = c$, $a_6 = d$, $a_7 = e$, $a_8 = qa^2/bcde$, $b_1 = \sqrt{a}$, $b_2 = -\sqrt{a}$, $b_3 = aq/b$, $b_4 = aq/c$, $b_5 = aq/d$, $b_6 = aq/e$, $b_7 = bcde/a$, $z = q$. Then the resulting ${}_8\Phi_7$ at the right-hand side can be summed using Bailey's nonterminating extension of Jackson's sum [5, equation (II.25)]. Thus, we obtain

$$
\begin{aligned}
&{}_9\Phi_8\left[\begin{matrix} aq, \sqrt{a}q^2, -\sqrt{a}q^2, bq, cq, dq, eq, a^2q^2/bcde, q; \\ \sqrt{a}q, -\sqrt{a}q, aq^2/b, aq^2/c, aq^2/d, aq^2/e, bcdeq/a, q^2; \end{matrix} q, q\right] \\
&\qquad = \frac{1}{q}\frac{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq/e)(1-bcde/a)(1-q)}{(1-q^2a)\,(1-b)(1-c)(1-d)(1-e)\,(1-qa^2/bcde)} \\
&\qquad \times\left(\frac{(aq, b/a, aq/cd, aq/ce, bde/a, aq/de, bce/a, bcd/a; q)_\infty}{(aq/c, aq/d, aq/e, bcde/a, bc/a, bd/a, be/a, aq/cde; q)_\infty} - 1\right. \\
&\qquad - \frac{(aq, c, d, e, a^2q/bcde, b/a, bq/c, bq/d, bq/e, b^2cde/a^2; q)_\infty}{(a/b, aq/c, aq/d, aq/e, bcde/a, bc/a, bd/a, be/a, aq/cde, b^2q/a; q)_\infty} \\
&\qquad \left.\times\, {}_8W_7\left(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{aq}{cde}; q, q\right)\right).
\end{aligned}
$$

Next we let $e \to 1$. The limit $\lim_{e\to 1}(1/(1-e))((\dots)_\infty/(\dots)_\infty - 1)$, which has to be computed at the right-hand side, is obtained by using l'Hôspital's rule, while the limit of the other expressions can be obtained directly. This results into

$$
\begin{aligned}
{}_9\Phi_8&\left[\begin{matrix}\sqrt{a}q^2, -\sqrt{a}q^2, bq, cq, dq, a^2q^2/bcd, aq, q, q;\\ \sqrt{a}q, -\sqrt{a}q, aq^2/b, aq^2/c, aq^2/d, bcdq/a, q^2, aq^2;\end{matrix} q, q\right]\\
&= \frac{1}{q}\frac{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq)(1-bcd/a)}{(1-q^2a)(1-b)(1-c)(1-d)(1-qa^2/bcd)}\\
&\times\left(\tilde{\psi}_q\left(\frac{aq}{d}\right) - \tilde{\psi}_q(aq) + \tilde{\psi}_q\left(\frac{aq}{c}\right) - \tilde{\psi}_q\left(\frac{aq}{cd}\right)\right.\\
&+ \tilde{\psi}_q\left(\frac{b}{a}\right) - \tilde{\psi}_q\left(\frac{bd}{a}\right) + \tilde{\psi}_q\left(\frac{bcd}{a}\right) - \tilde{\psi}_q\left(\frac{bc}{a}\right)\\
&- \frac{(1-q)\left(c, d, q, a^2q/bcd, bq/c, bq/d, bq, b^2cd/a^2; q\right)_\infty}{(a/b, aq/c, aq/d, bcd/a, bc/a, bd/a, aq/cd, b^2q/a; q)_\infty}\\
&\left.\times\, {}_8W_7\left(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{b}{a}, \frac{aq}{cd}; q, q\right)\right).
\end{aligned}
\tag{4.1}
$$

Replacing a by q^a, b by q^b, c by q^c, d by q^d, and letting $q \uparrow 1$, it is seen that identity (4.1) is a nonterminating q-analogue of [4, equation (1.3)]. In fact, by use of (2.17), we thus get

$$
\begin{aligned}
&{}_8F_7\left[\begin{matrix}2+a/2, b+1, c+1, d+1, 2+2a-b-c-d, a+1, 1, 1;\\ 1+a/2, 2+a-b, 2+a-c, 2+a-d, 1+b+c+d-a, 2, 2+a;\end{matrix} 1\right]\\
&= \frac{(1+a-b)(1+a-c)(1+a-d)(1+a)(b+c+d-a)}{(2+a)\,bcd\,(1+2a-b-c-d)}\\
&\times\Big(\psi(1+a-d) - \psi(1+a) + \psi(1+a-c) - \psi(1+a-c-d)\\
&+ \psi(b-a) - \psi(b+d-a) + \psi(b+c+d-a) - \psi(b+c-a)\\
&- \Gamma\left[\begin{matrix}a-b, 1+a-c, 1+a-d, b+c+d-a, b+c-a, b+d-a, 1+a-c-d\\ c, d, 1+2a-b-c-d, 1+b-c, 1+b-d, 1+b\end{matrix}\right.\\
&\qquad\left.\begin{matrix}1+2b-a\\ 2b+c+d-2a\end{matrix}\right]{}_7V_6(2b-a; b, b+c-a, b+d-a, b-a, 1+a-c-d; 1)\Big),
\end{aligned}
\tag{4.2}
$$

a nonterminating extension of [4, equation (1.3)].

Next we find q-analogues for [4, equations (3.1)–(3.6)]. First, in (4.1), set $d = q^{-N}$ and then let $N \to \infty$. Observing that with $d = q^{-N}$ the last term in (4.1) vanishes, by means of (2.19) we get

$$
\begin{aligned}
{}_7\Phi_6&\left[\begin{matrix}aq, \sqrt{a}q^2, -\sqrt{a}q^2, bq, cq, q, q;\\ \sqrt{a}q, -\sqrt{a}q, aq^2/b, aq^2/c, aq^2, q^2;\end{matrix} q, \frac{qa}{bc}\right]\\
&= \frac{bc}{qa}\frac{(1-aq/b)(1-aq/c)(1-aq)}{(1-q^2a)(1-b)(1-c)}\left(\tilde{\psi}_q\left(\frac{aq}{c}\right) - \tilde{\psi}_q(aq) + \tilde{\psi}_q\left(\frac{aq}{b}\right) - \tilde{\psi}_q\left(\frac{aq}{bc}\right)\right).
\end{aligned}
\tag{4.3}
$$

This is a q-analogue of [4, equation (3.2)], of which [4, equation (3.1)] is a terminating special case. Letting $c \to \infty$ in (4.3), we obtain a q-analogue of [4, equation (3.3)],

$$
{}_6\Phi_6\left[\begin{matrix}aq, \sqrt{a}q^2, -\sqrt{a}q^2, bq, q, q;\\ \sqrt{a}q, -\sqrt{a}q, aq^2/b, aq^2, q^2, 0;\end{matrix} q, \frac{q^2a}{b}\right] = \frac{b}{qa}\frac{(1-aq/b)(1-aq)}{(1-q^2a)(1-b)}\left(\tilde{\psi}_q(aq) - \tilde{\psi}_q\left(\frac{aq}{b}\right)\right). \tag{4.4}
$$

For $c = \sqrt{a}$, (4.3) reduces to a q-analogue of [4, equation (3.4)],

$$
\begin{aligned}
{}_5\Phi_4&\left[\begin{matrix}aq, -\sqrt{a}q^2, bq, q, q;\\ -\sqrt{a}q, aq^2/b, aq^2, q^2;\end{matrix} q, \frac{q\sqrt{a}}{b}\right]\\
&= \frac{b}{q\sqrt{a}}\frac{(1-aq/b)(1-aq)(1+\sqrt{a})}{(1-b)(1-a)(1+q\sqrt{a})}\left(\tilde{\psi}_q\left(\sqrt{a}q\right) - \tilde{\psi}_q(aq) + \tilde{\psi}_q\left(\frac{aq}{b}\right) - \tilde{\psi}_q\left(\frac{q\sqrt{a}}{b}\right)\right).
\end{aligned}
\tag{4.5}
$$

Similarly, setting $b=\sqrt{a}$ in (4.5) yields

$$ {}_4\Phi_4\left[\begin{matrix}aq,-\sqrt{a}q^2,q,q;\\ -\sqrt{a}q,aq^2,q^2,0;\end{matrix}q,q^2\sqrt{a}\right]=\frac{1}{q\sqrt{a}}\frac{(1-aq)(1+\sqrt{a})}{(1-a)(1+q\sqrt{a})}\left(\tilde{\psi}_q(aq)-\tilde{\psi}_q\left(q\sqrt{a}\right)\right), \tag{4.6}$$

a q-analogue for [4, equation (3.5)]. A q-analogue for [4, equation (3.6)] could be obtained by starting with the duplication formula for the q-gamma function [5, equation (1.10.9)].

As in the previous section, it should be observed that some identities could have been obtained in an alternative way. For instance, (4.3) follows when applying the nonterminating Watson transformation [5, Appendix, (III.36)] to

$$ {}_8W_7\left(aq^2;cq,q,bq,q,aq;q,\frac{aq}{bc}\right)$$

and, with the help of (2.13b), writing the resulting ${}_3\Phi_2$'s in terms of q-digamma functions. Also we remark that all the series in this section are very well-poised series with cancelling leading parameter. Thus, the ${}_9\Phi_8$ in (4.1) actually is the very well-poised series

$$ {}_{10}W_9\left(aq^2;aq,bq,cq,dq,\frac{a^2q^2}{bcd};q,q\right).$$

5. IDENTITIES RELATED TO VERMA AND JAIN'S q-ANALOGUE OF WHIPPLE'S ${}_3F_2$-SUM

In this section, we rely on Verma and Jain's summation [11, equation (1.6)] for a very well-poised ${}_8\Phi_7$,

$$\begin{aligned} &{}_8\Phi_7\left[\begin{matrix}-c,iq\sqrt{c},-iq\sqrt{c},a,q/a,c,-d,-q/d;\\ i\sqrt{c},-i\sqrt{c},-cq/a,-ac,-q,cq/d,cd;\end{matrix}q,c\right]\\ &\qquad=\frac{(-c,-cq;q)_\infty\left(acd,acq/d,cdq/a,cq^2/ad;q^2\right)_\infty}{(cd,cq/d,-ac,-cq/a;q)_\infty}. \end{aligned}\tag{5.1}$$

We also use two summations contiguous to (5.1), namely

$$\begin{aligned} &{}_8\Phi_7\left[\begin{matrix}-c,iq\sqrt{c},-iq\sqrt{c},a,1/a,c,-d,-q/d;\\ i\sqrt{c},-i\sqrt{c},-cq/a,-acq,-q,cq/d,cd;\end{matrix}q,cq\right]\\ &\qquad=\frac{1}{1+a}\frac{(-cq,-cq;q)_\infty\left(acd,acq/d,cdq/a,cq^2/ad;q^2\right)_\infty}{(cd,cq/d,-acq,-cq/a;q)_\infty}\\ &\qquad\quad+\frac{a}{1+a}\frac{(-cq,-cq;q)_\infty\left(acdq,acq^2/d,cd/a,cq/ad;q^2\right)_\infty}{(cd,cq/d,-acq,-cq/a;q)_\infty} \end{aligned}\tag{5.2}$$

and

$$\begin{aligned} &{}_8\Phi_7\left[\begin{matrix}-cq,i\sqrt{c}q^{3/2},-i\sqrt{c}q^{3/2},aq,q/a,c,-dq,-q^2/d;\\ i\sqrt{cq},-i\sqrt{cq},-cq/a,-acq,-q^2,cq/d,cd;\end{matrix}q,\frac{c}{q}\right]\\ &\qquad=\frac{ad(1+c/aq)(1+q)}{c(1-a)(1-c/q)(1+a)(1+d)(1+d/q)}\\ &\qquad\quad\times\frac{\left(-c,-cq^2;q\right)_\infty\left(acd/q,ac/d,cd/a,cq/ad;q^2\right)_\infty}{(cd,cq/d,-acq,-cq/a;q)_\infty}\\ &\qquad\quad-\frac{ad(1+ac/q)(1+q)}{c(1-a)(1-c/q)(1+a)(1+d)(1+d/q)}\\ &\qquad\quad\times\frac{\left(-c,-cq^2;q\right)_\infty\left(acd,acq/d,cd/aq,c/ad;q^2\right)_\infty}{(cd,cq/d,-acq,-cq/a;q)_\infty}. \end{aligned}\tag{5.3}$$

Identity (5.2) is a q-analogue of [3, equation (3)]. It follows from the contiguous relation

$$\begin{aligned}{}_{r+1}\Phi_r\left[\begin{matrix}x,y,(A);\\u/x,u/y,(B);\end{matrix}q,z\right] &= \frac{q\,x\,y(1-x)(1-u/xq)}{u\,(x-y)(1-xyq/u)}\;{}_{r+1}\Phi_r\left[\begin{matrix}xq,y,(A);\\u/xq,u/y,(B);\end{matrix}q,\frac{z}{q}\right]\\&\quad-\frac{q\,x\,y(1-y)(1-u/yq)}{u\,(x-y)(1-xyq/u)}\;{}_{r+1}\Phi_r\left[\begin{matrix}x,yq,(A);\\u/x,u/yq,(B);\end{matrix}q,\frac{z}{q}\right].\end{aligned}\tag{5.4}$$

In fact, by setting $r=7$, $x=a$, $y=1/a$, $u=-c$, and (A), (B), z such that the left-hand side of (5.4) becomes the ${}_8\Phi_7$ in (5.2), at the right-hand side of (5.4) we obtain two ${}_8\Phi_7$'s which can be evaluated by means of (5.1).

Setting $c=1$ in (5.2) and replacing a by $\sqrt{ABq}$ and d by $\sqrt{Aq/B}$ leads to a three-product relation,

$$\begin{aligned}&(A\,q;q^2)_\infty\,(B\,q;q^2)_\infty\left(\frac{q}{A};q^2\right)_\infty\left(\frac{q}{B};q^2\right)_\infty\\&\quad+\sqrt{A\,B\,q}\left(\frac{1}{A};q^2\right)_\infty(A\,q^2;q^2)_\infty\left(\frac{1}{B};q^2\right)_\infty(B\,q^2;q^2)_\infty\\&=\frac{(-\sqrt{A\,B\,q};q)_\infty\left(-\sqrt{q/AB};q\right)_\infty\left(\sqrt{Aq}/\sqrt{B};q\right)_\infty\left(\sqrt{Bq}/\sqrt{A};q\right)_\infty}{(-q;q)_\infty{}^2},\end{aligned}\tag{5.5}$$

which will be used later. Written in the form

$$\begin{aligned}&\left(Aq,\frac{q}{A},Bq,\frac{q}{B},-1,-q^2,-q,-q;q^2\right)_\infty+\sqrt{ABq}\left(Aq^2,\frac{1}{A},Bq^2,\frac{1}{B},-1,-q^2,-q,-q;q^2\right)_\infty\\&=2\cdot\left(-q^{1/2}\sqrt{AB},-\frac{q^{3/2}}{\sqrt{AB}},-q^{3/2}\sqrt{AB},-\frac{q^{1/2}}{\sqrt{AB}},q^{1/2}\sqrt{\frac{A}{B}},q^{3/2}\sqrt{\frac{B}{A}},q^{3/2}\sqrt{\frac{A}{B}},q^{1/2}\sqrt{\frac{B}{A}};q^2\right)_\infty,\end{aligned}\tag{5.6}$$

it turns out to be a special case of a relation connecting four infinite products due to Slater [5, Example 5.22; 20, equation (3)]. In fact, replacing q by q^2, a by $\sqrt{ABq^3}$, b by $\sqrt{ABq}$, c by $-q$, d by $-q^2$, e by Aq^2, f by Bq^2, g by 1, and h by q in Slater's formula gives (5.6) after little manipulation.

Identity (5.3) is a q-analogue of [3, equation (4)]. It is derived from the contiguous relation

$$\begin{aligned}&{}_{r+1}\Phi_r\left[\begin{matrix}x,y,u/q,(A);\\u/x,u/y,u/q^2,(B);\end{matrix}q,z\right]\\&\quad=\frac{(1-u/xq^2)(1-u/yq^2)(1-u/xq)(1-u/yq)}{(1-u/q^2)(-x/q+y/q)(1-u/xyq)z}\,\frac{\prod_{i=1}^{r-3}(1-B_i/q)}{\prod_{i=1}^{r-2}(1-A_i/q)}\\&\qquad\times{}_r\Phi_{r-1}\left[\begin{matrix}x/q,y,(A/q);\\u/xq,u/yq^2,(B/q);\end{matrix}q,z\right]\\&\qquad-\frac{\left(1-u/xq^2\right)\left(1-u/yq^2\right)(1-u/xq)(1-u/yq)}{(1-u/q^2)\,(-x/q+y/q)(1-u/xyq)z}\,\frac{\prod_{i=1}^{r-3}(1-B_i/q)}{\prod_{i=1}^{r-2}(1-A_i/q)}\\&\qquad\times{}_r\Phi_{r-1}\left[\begin{matrix}x,y/q,(A/q);\\u/xq^2,u/yq,(B/q);\end{matrix}q,z\right].\end{aligned}\tag{5.7}$$

In order to see this, set $x=aq$, $y=q/a$, $u=-cq^2$, $(A)=(-c,i\sqrt{c}q^{3/2},-i\sqrt{c}q^{3/2},c,-dq,-q^2/d)$, $(B)=(i\sqrt{cq},-i\sqrt{cq},-q^2,cq/d,cd)$ in (5.7). Thus, since the parameter $-c$ is cancelling, at the left-hand side we just obtain the ${}_8\Phi_7$ of (5.3). At the right-hand side we obtain two ${}_8\Phi_7$'s which can be summed by means of (5.1).

As first application of our method in this section, in (1.1) choose $a_1=-c$, $a_2=i\sqrt{c}\,q$, $a_3=-i\sqrt{c}\,q$, $a_4=a$, $a_5=q/a$, $a_6=c$, $a_7=-d$, $a_8=-q/d$, $b_1=i\sqrt{c}$, $b_2=-i\sqrt{c}$, $b_3=-c\,q/a$,

$b_4 = -a\,c$, $b_5 = -q$, $b_6 = c\,q/d$, $b_7 = c\,d$, $z = c$. Then the resulting ${}_8\Phi_7$ at the right-hand side can be summed using (5.1). Letting $a \to 1$ with the help of l'Hôspital's rule we obtain the identity

$$\begin{aligned}
&{}_7\Phi_6\left[\begin{matrix} i\sqrt{c}\,q^2, -i\sqrt{c}\,q^2, q, c\,q, -d\,q, -q^2/d, q; \\ i\sqrt{c}\,q, -i\sqrt{c}\,q, -c\,q^2, -q^2, c\,q^2/d, c\,d\,q; \end{matrix} q, c\right] \\
&\qquad = \frac{(1-c\,d)\,(1-c\,q/d)\,(1+c)\,(1+q)\,(1+c\,q)}{c\,(1-c)\,(1-q)\,(1+d)\,(1+q/d)\,(1+c\,q^2)} \\
&\qquad\quad \times \left(\frac{c(1-q)}{(1+c)} + \frac{\tilde{\psi}_{q^2}(c\,d\,q) - \tilde{\psi}_{q^2}(c\,d) + \tilde{\psi}_{q^2}(c\,q^2/d) - \tilde{\psi}_{q^2}(c\,q/d)}{1+q}\right).
\end{aligned} \tag{5.8}$$

Replacing c by q^c, d by q^d and letting $q \uparrow 1$, by (2.17) we see that this is a q-analogue of [2, equation (1)]. On the other hand, by replacing c by $-q^c$, d by $-q^d$ and letting $q \uparrow 1$, by (2.17) we arrive at the ${}_5F_4$-evaluation

$$\begin{aligned}
&{}_5F_4\left[\begin{matrix} 2+c/2, 1, 1+d, 2-d, 1; \\ 1+c/2, 2+c, 2+c-d, 1+c+d; \end{matrix} -1\right] = \frac{c(1+c)(1+c-d)(c+d)}{(2+c)(1-d)d} \\
&\qquad \times \left(\frac{1}{c} + \frac{1}{2}\left(\psi\left(\frac{1+c-d}{2}\right) - \psi\left(\frac{2+c-d}{2}\right) + \psi\left(\frac{c+d}{2}\right) - \psi\left(\frac{1+c+d}{2}\right)\right)\right).
\end{aligned} \tag{5.9}$$

The series in (5.8) actually is the very well-poised series

$${}_8W_7\left(-c\,q^2; q, c\,q, -d\,q, -\frac{q^2}{d}, q; q, c\right),$$

where the parameter $-cq^2$ is cancelling. Applying the transformation [5, Appendix, (III.24)] to this very well-poised series we find

$$\begin{aligned}
&{}_8\Phi_7\left[\begin{matrix} -c^2q, icq^{3/2}, -icq^{3/2}, -q, cq/d, cd, -cq, c; \\ ic\sqrt{q}, -ic\sqrt{q}, c^2q, -cdq, -cq^2/d, cq, -cq^2; \end{matrix} q, q\right] \\
&\qquad = \frac{(1+c)\,(1+cq)}{c\,(1-q)\,(1+d)(1+q/d)}\,\frac{(cd, cq/d, -q, -c^2q^2; q)_\infty}{(q, c^2q, -cdq, -cq^2/d; q)_\infty} \\
&\qquad\quad \times \left(\frac{c(1-q)}{1+c} + \frac{\tilde{\psi}_{q^2}(cdq) - \tilde{\psi}_{q^2}(cd) + \tilde{\psi}_{q^2}(cq^2/d) - \tilde{\psi}_{q^2}(cq/d)}{1+q}\right).
\end{aligned} \tag{5.10}$$

This identity is a q-analogue of an identity of Watson [2, equation (6); 21].

Next in (1.1) choose $a_1 = -c$, $a_2 = i\,q\sqrt{c}$, $a_3 = -i\,q\sqrt{c}$, $a_4 = a$, $a_5 = 1/a$, $a_6 = c$, $a_7 = -d$, $a_8 = -q/d$, $b_1 = i\sqrt{c}$, $b_2 = -i\sqrt{c}$, $b_3 = -c\,q/a$, $b_4 = -a\,c\,q$, $b_5 = -q$, $b_6 = c\,q/d$, $b_7 = c\,d$, $z = cq$. After applying (5.2) at the right-hand side, this yields

$$\begin{aligned}
&{}_9\Phi_8\left[\begin{matrix} -c\,q, i\sqrt{c}\,q^2, -i\sqrt{c}\,q^2, q/a, a\,q, c\,q, -d\,q, -q^2/d, q; \\ i\sqrt{c}\,q, -i\sqrt{c}\,q, -a\,c\,q^2, -c\,q^2/a, -q^2, c\,q^2/d, c\,d\,q, q^2; \end{matrix} q, c\,q\right] \\
&\qquad = c\,\frac{(1-c\,d)\,(1-d/cq)\,(1-q)\,(1+q)\,(1+a/cq)\,(1+a\,c\,q)}{(1-a)^2\,(1-c)\,(1+d)\,(1+d/q)\,(1+c\,q^2)} \\
&\qquad\quad \times \left(\frac{a}{1+a}\,\frac{(-c\,q, -c\,q; q)_\infty\,(c\,d/a, c\,q/a\,d, a\,c\,d\,q, a\,c\,q^2/d; q^2)_\infty}{(c\,d, -c\,q/a, -a\,c\,q, c\,q/d; q)_\infty}\right. \\
&\qquad\quad \left. + \frac{1}{1+a}\,\frac{(-c\,q, -c\,q; q)_\infty\,(a\,c\,d, a\,c\,q/d, c\,d\,q/a, c\,q^2/a\,d; q^2)_\infty}{(c\,d, -c\,q/a, -a\,c\,q, c\,q/d; q)_\infty} - 1\right).
\end{aligned} \tag{5.11}$$

Replacing d by cq/d and letting $c \to 1$, we get a q-analogue of [2, equation (2)], which after some manipulation can be brought into the form

$$
\begin{aligned}
{}_9\Phi_8&\left[\begin{matrix} i\,q^2, -i\,q^2, q/a, a\,q, q, -q^2/d, -d\,q, -q, q; \\ i\,q, -i\,q, -a\,q^2, -q^2/a, -q^2, d\,q, q^2/d, q^2; \end{matrix} q, q\right] \\
&= \frac{(1-d)\,(1-d/q)\,(1+q)\,(1+a/q)\,(1+a\,q)}{(1-a)^2\,(1+d)\,(1+d/q)\;(1+q^2)} \\
&\times \left(2\,\tilde\psi_q\left(\frac{q}{d}\right) + \tilde\psi_q\left(-\frac{q}{a}\right) + \tilde\psi_q(-a\,q) - 2\,\tilde\psi_q(-q) - 2\frac{(-q,-q^2;q)_\infty}{(d,q/d,-q/a,-a;q)_\infty}\right. \\
&\times \left(\left(\frac{a\,q}{d}, a\,d, \frac{q^2}{a\,d}, \frac{d\,q}{a}; q^2\right)_\infty \left(\tilde\psi_{q^2}\left(\frac{a\,q}{d}\right) + \tilde\psi_{q^2}\left(\frac{q^2}{a\,d}\right)\right)\right. \\
&\left.\left. + a\left(\frac{q}{a\,d}, \frac{d}{a}, \frac{a\,q^2}{d}, a\,d\,q; q^2\right)_\infty \left(\tilde\psi_{q^2}\left(\frac{q}{a\,d}\right) + \tilde\psi_{q^2}\left(\frac{a\,q^2}{d}\right)\right)\right)\right).
\end{aligned}
\tag{5.12}
$$

In the computations which finally lead to (5.11) we also used the three-product relation (5.5).

An interesting special case of (5.12) arises when setting $d = aq^{-2N}$ and then letting $N \to \infty$. By this procedure, the infinite products at the right-hand side approach to a definite limit while the $\tilde\psi_{q^2}$'s vanish. Hence, what remains is

$$
\begin{aligned}
{}_7\Phi_6&\left[\begin{matrix} i\,q^2, -i\,q^2, q/a, a\,q, q, -q, q; \\ i\,q, -i\,q, -a\,q^2, -q^2/a, -q^2, q^2; \end{matrix} q, -q\right] \\
&= \frac{(1+a/q)\,(1+q)\,(1+a\,q)}{(1-a)^2\;(1+q^2)}\left(\tilde\psi_q\left(-\frac{q}{a}\right) + \tilde\psi_q(-a\,q) - 2\,\tilde\psi_q(-q)\right),
\end{aligned}
\tag{5.13}
$$

an identity which has no reasonable counterpart for $q \uparrow 1$.

If in (5.11) we let $a \to 1$, by using l'Hôspital's rule twice and employing (2.25) we obtain

$$
\begin{aligned}
{}_9\Phi_8&\left[\begin{matrix} i\sqrt{c}q^2, -i\sqrt{c}q^2, q, q, cq, -dq, -q^2/d, -c\,q, q; \\ i\sqrt{c}q, -i\sqrt{c}q, -cq^2, -cq^2, -q^2, cq^2/d, cdq, q^2; \end{matrix} q, cq\right] \\
&= \frac{(1-cd)(1-d/cq)(1+q)(1+cq)^2}{2\,q(1-c)(1-q)(1+d)(1+d/q)\,(1+cq^2)} \\
&\left(\frac{1}{(1+q)^2}\left(\tilde\psi_{q^2}(cd) + \tilde\psi_{q^2}\left(\frac{cq}{d}\right) - \tilde\psi_{q^2}(cdq) - \tilde\psi_{q^2}\left(\frac{cq^2}{d}\right)\right)^2 - \tilde\psi_q^{(1)}(cd) - \tilde\psi_q^{(1)}\left(\frac{cq}{d}\right)\right. \\
&\left. -\frac{(1-q)}{1+q}\left(\tilde\psi_{q^2}(cd) + \tilde\psi_{q^2}\left(\frac{cq}{d}\right) - \tilde\psi_{q^2}(cdq) - \tilde\psi_{q^2}\left(\frac{cq^2}{d}\right)\right) + 2\tilde\psi_q^{(1)}(-cq)\right).
\end{aligned}
\tag{5.14}
$$

This is a q-analogue of [2, equation (3)], which is seen by replacing c by q^c, d by q^d, and letting $q \uparrow 1$. For performing the limit, we have to take into account (2.17), (2.22), and (2.23).

On the other hand, replacing c by $-q^c$ and d by $-q^d$ in (5.14) gives

$$
\begin{aligned}
{}_7F_6&\left[\begin{matrix} 2+\frac{c}{2}, 1, 1, 1+d, 2-d, 1+c, 1; \\ 1+\frac{c}{2}, 2+c, 2+c, 2+c-d, 1+c+d, 2; \end{matrix} -1\right] \\
&= \frac{(1+c)^2(d-c-1)(c+d)}{2\,d\,(2+c)(d-1)}\left(\frac{1}{4}\left(\psi\left(\frac{1+c-d}{2}\right) - \psi\left(\frac{2+c-d}{2}\right) + \psi\left(\frac{c+d}{2}\right)\right.\right. \\
&\left.\left. -\,\psi\left(\frac{1+c+d}{2}\right)\right)^2 + 2\psi^{(1)}(1+c) - \psi^{(1)}(1+c-d) - \psi^{(1)}(c+d)\right).
\end{aligned}
\tag{5.15}
$$

An interesting special case of (5.14) is obtained by setting $d = q$,

$$
{}_7\Phi_6\left[\begin{matrix} i\sqrt{c}q^2, -i\sqrt{c}q^2, -q, -c\,q, q, q, q; \\ i\sqrt{c}q, -i\sqrt{c}q, cq^2, q^2, -cq^2, -cq^2; \end{matrix} q, cq\right] = \frac{(1-cq)(1+cq)^2}{2\,c(1-q)q\,(1+cq^2)}\left(\tilde\psi_q^{(1)}(cq) - \tilde\psi_q^{(1)}(-cq)\right).
\tag{5.16}
$$

This identity is a q-analogue of [2, second identity on p. 216]. To obtain another identity from (5.14), we replace c by $-c$, set $d=-q^{-N}$ and let $N\to\infty$. By using (2.19) and (2.24) we find that

$$
{}_7\Phi_6\left[\begin{matrix}\sqrt{c}\,q^2,-\sqrt{c}\,q^2,q,q,-c\,q,c\,q,q;\\ \sqrt{c}\,q,-\sqrt{c}\,q,c\,q^2,c\,q^2,-q^2,q^2;\end{matrix}q,-q\right]
= \frac{(1+q)\,(1-c\,q)^2}{2\,q\,(1+c)\,(1-q)\,(1-c\,q^2)}\Bigg(\frac{\left(\tilde\psi_{q^2}(c)+\tilde\psi_{q^2}(q/c)-\tilde\psi_{q^2}(c\,q)-\tilde\psi_{q^2}\left(q^2/c\right)\right)^2}{(1+q)^2}
+\tilde\psi_q^{(1)}(c\,q)-\tilde\psi_q^{(1)}\left(\frac{1}{c}\right)-\frac{(1-q)}{1+q}\left(\tilde\psi_{q^2}(c)+\tilde\psi_{q^2}\left(\frac{q}{c}\right)-\tilde\psi_{q^2}(c\,q)-\tilde\psi_{q^2}\left(\frac{q^2}{c}\right)\right)\Bigg).
\tag{5.17}
$$

Replacing c by q^c and letting $q\uparrow 1$ gives an evaluation for a ${}_5F_4$,

$$
{}_5F_4\left[\begin{matrix}2+\frac{c}{2},1,1,1+c,1;\\ 1+\frac{c}{2},2+c,2+c,2;\end{matrix}-1\right]
=\frac{(c+1)^2}{2\,(c+2)}\left(\frac{1}{4}\left(\psi\left(\frac{1-c}{2}\right)-\psi\left(1-\frac{c}{2}\right)+\psi\left(\frac{c}{2}\right)-\psi\left(\frac{1+c}{2}\right)\right)^2+\psi^{(1)}(1+c)-\psi^{(1)}(-c)\right).
\tag{5.18}
$$

Finally, in (1.1) choose $a_1=-c\,q$, $a_2=i\sqrt{c}\,q^{3/2}$, $a_3=-i\sqrt{c}\,q^{3/2}$, $a_4=a\,q$, $a_5=q/a$, $a_6=c$, $a_7=-d\,q$, $a_8=-q^2/d$, $b_1=i\sqrt{c\,q}$, $b_2=-i\sqrt{c\,q}$, $b_3=-c\,q/a$, $b_4=-a\,c\,q$, $b_5=-q^2$, $b_6=c\,q/d$, $b_7=c\,d$, $z=c/q$. Then the resulting ${}_8\Phi_7$ at the right-hand side can be summed using (5.3). Upon letting $a\to q$ we obtain the following q-analogue of [3, third identity on p. 272]

$$
{}_9\Phi_8\left[\begin{matrix}i\sqrt{c}\,q^{5/2},-i\sqrt{c}\,q^{5/2},c\,q,-d\,q^2,-q^3/d,-c\,q^2,q^3,q,q;\\ i\sqrt{c}\,q^{3/2},-i\sqrt{c}\,q^{3/2},-q^3,c\,q^2/d,c\,d\,q,q^2,-c\,q,-c\,q^3;\end{matrix}q,\frac{c}{q}\right]
=\frac{q\,(1-cd)(1-d/cq)(1+c)\,(1+q^2)\,(1+cq^2)}{(1-q)(1-c)\,(1+d/q^2)\,(1+dq)(1+q)\,(1+cq^3)}
\left(-\frac{(1+c/q^2)\,((1-d)+(d/q)(1-c)+(d^2/q)(1-c/d))}{(1-c/q)(1+d)(1+d/q)}+\frac{1-q}{1+q}-\frac{c\,(1-q)}{q(1+c)}-\frac{c\,(1-q)}{1+cq}
-\frac{1}{q(1+q)}\left(\frac{2\,d\,q\,(1-c/d)(1-cd/q)\,(1+c/q^2)}{c(1-c/q)(1-q)(1+d)(1+d/q)}-1\right)
\left(\tilde\psi_{q^2}(c\,d)-\tilde\psi_{q^2}(c\,d\,q)+\tilde\psi_{q^2}\left(\frac{c\,q}{d}\right)-\tilde\psi_{q^2}\left(\frac{c\,q^2}{d}\right)\right)\right).
\tag{5.19}
$$

Replacing c by $-q^c$, d by $-q^d$ and letting $q\uparrow 1$ leads to an evaluation for a ${}_7F_6$,

$$
{}_7F_6\left[\begin{matrix}\frac{5}{2}+\frac{c}{2},2+d,3-d,2+c,3,1,1;\\ \frac{3}{2}+\frac{c}{2},2+c-d,1+c+d,2,1+c,3+c;\end{matrix}-1\right]
=\frac{c(2+c)(d-c-1)(c+d)}{2(3+c)(d-2)(1+d)}\left(\frac{1}{c}+\frac{1}{1+c}+\frac{(c-2)(2d-c-1)}{(d-1)d}
-\left(\frac{(c-2)(c-d)(c+d-1)}{2(d-1)d}-1\right)
\times\left(\psi\left(\frac{1+c-d}{2}\right)-\psi\left(\frac{2+c-d}{2}\right)+\psi\left(\frac{c+d}{2}\right)-\psi\left(\frac{1+c+d}{2}\right)\right)\right).
\tag{5.20}
$$

As before in (5.14), it is of particular interest to set $d=q$ in (5.19). With this choice of d, (5.19) eventually reduces to

$$
{}_7\Phi_6\left[\begin{matrix} i\sqrt{c}q^{5/2}, -i\sqrt{c}q^{5/2}, q^3, q, -q^2, -cq^2, q; \\ i\sqrt{c}q^{3/2}, -i\sqrt{c}q^{3/2}, -cq, -cq^3, cq^2, q^2; \end{matrix} q, \frac{c}{q}\right] = \frac{(1-cq)\left(1-2c-c^2+q+2cq-c^2q\right)\left(1+cq^2\right)}{(1-c)(1-c/q)(1+q)(1+cq)\left(1+cq^3\right)}. \tag{5.21}
$$

Replacing c by q^c and $-q^c$, and then letting q tend to 1^- in (5.21), produces the identities

$$
{}_3F_2\left[\begin{matrix} 3, 1, 1; \\ 2+c, 2; \end{matrix} 1\right] = \frac{(1+c)(2c-1)}{2(c-1)c} \tag{5.22}
$$

and

$$
{}_5F_4\left[\begin{matrix} 2+c, \frac{5}{2}+\frac{c}{2}, 3, 1, 1; \\ \frac{3}{2}+\frac{c}{2}, 1+c, 3+c, 2; \end{matrix} -1\right] = \frac{(2+c)(1+2c)}{2(1+c)(3+c)}, \tag{5.23}
$$

respectively.

The last remark at the end of Section 4 also applies here. All the series appearing in (5.8)–(5.23) with the exception of (5.22) are very well-poised series with cancelling leading parameter.

6. IDENTITIES RELATED TO VERMA AND JAIN'S q-ANALOGUE OF WATSON'S ${}_3F_2$-SUM

In this section, we rely on Verma and Jain's identity [11, equation (1.5)]

$$
\begin{aligned}
&{}_8\Phi_7\left[\begin{matrix} -\sqrt{ab}\,c/\sqrt{q}, i(ab)^{1/4}\sqrt{c}\,q^{3/4}, -i(ab)^{1/4}\sqrt{c}\,q^{3/4}, a, b, \\ i(ab)^{1/4}\sqrt{c}/q^{1/4}, -i(ab)^{1/4}\sqrt{c}/q^{1/4}, -c\sqrt{bq}/\sqrt{a}, -c\sqrt{aq}/\sqrt{b}, \end{matrix}\right. \\
&\qquad\qquad\qquad\qquad \left.\begin{matrix} c, -c, -\sqrt{abq}/c; \\ -\sqrt{abq}, \sqrt{abq}, c^2; \end{matrix} q, \frac{c\sqrt{q}}{\sqrt{ab}}\right] \\
&\quad = \frac{\left(-c\sqrt{abq}, -c\sqrt{q}/\sqrt{ab}; q\right)_\infty}{\left(-c\sqrt{bq}/\sqrt{a}, -c\sqrt{aq}/\sqrt{b}; q\right)_\infty} \frac{\left(aq, bq, c^2q/a, c^2q/b; q^2\right)_\infty}{\left(q, abq, c^2q, c^2q/ab; q^2\right)_\infty}.
\end{aligned} \tag{6.1}
$$

We start by recording two identities contiguous to (6.1),

$$
\begin{aligned}
&{}_8\Phi_7\left[\begin{matrix} -\sqrt{ab}\,c/\sqrt{q}, i(ab)^{1/4}\sqrt{c}\,q^{3/4}, -i(ab)^{1/4}\sqrt{c}\,q^{3/4}, a, b, \\ i(ab)^{1/4}\sqrt{c}/q^{1/4}, -i(ab)^{1/4}\sqrt{c}/q^{1/4}, -c\sqrt{bq}/\sqrt{a}, -c\sqrt{aq}/\sqrt{b}, \end{matrix}\right. \\
&\qquad\qquad\qquad\qquad \left.\begin{matrix} cq, -c, -\sqrt{ab}/c\sqrt{q}; \\ -\sqrt{ab}/\sqrt{q}, \sqrt{abq}, c^2q; \end{matrix} q, \frac{c\sqrt{q}}{\sqrt{ab}}\right] \\
&\quad = \frac{\left(1+c^2\sqrt{q}/\sqrt{ab}\right)}{\left(1+\sqrt{ab}/\sqrt{q}\right)} \frac{\left(-c\sqrt{abq}, -c\sqrt{q}/\sqrt{ab}; q\right)_\infty}{\left(-c\sqrt{bq}/\sqrt{a}, -c\sqrt{aq}/\sqrt{b}; q\right)_\infty} \frac{\left(a, b, c^2q^2/a, c^2q^2/b; q^2\right)_\infty}{\left(q, abq, c^2q, c^2q/ab; q^2\right)_\infty} \\
&\qquad + \frac{\left(-c\sqrt{abq}, -c\sqrt{q}/\sqrt{ab}; q\right)_\infty}{\left(-c\sqrt{bq}/\sqrt{a}, -c\sqrt{aq}/\sqrt{b}; q\right)_\infty} \frac{\left(aq, bq, c^2q/a, c^2q/b; q^2\right)_\infty}{\left(q, abq, c^2q, c^2q/ab; q^2\right)_\infty}
\end{aligned} \tag{6.2}
$$

and

$$
\begin{aligned}
&{}_8\Phi_7\left[\begin{matrix} -\sqrt{ab}\,c/\sqrt{q}, i(ab)^{1/4}\sqrt{c}\,q^{3/4}, -i(ab)^{1/4}\sqrt{c}\,q^{3/4}, a, b, \\ i(ab)^{1/4}\sqrt{c}/q^{1/4}, -i(ab)^{1/4}\sqrt{c}/q^{1/4}, -c\sqrt{bq}/\sqrt{a}, -c\sqrt{aq}/\sqrt{b}, \end{matrix}\right. \\
&\qquad\qquad\qquad\qquad \left.\begin{matrix} c, -c, -\sqrt{ab}/c\sqrt{q}; \\ -\sqrt{abq}, \sqrt{abq}, c^2q; \end{matrix} q, \frac{cq^{3/2}}{\sqrt{ab}}\right] \\
&\quad = \frac{\left(-c\sqrt{abq}, -c\sqrt{q}/\sqrt{ab}; q\right)_\infty}{\left(-c\sqrt{bq}/\sqrt{a}, -c\sqrt{aq}/\sqrt{b}; q\right)_\infty} \frac{\left(aq, bq, c^2q/a, c^2q/b; q^2\right)_\infty}{\left(q, abq, c^2q, c^2q/ab; q^2\right)_\infty} \\
&\qquad - \frac{c\sqrt{q}}{\sqrt{ab}} \frac{\left(-c\sqrt{abq}, -c\sqrt{q}/\sqrt{ab}; q\right)_\infty}{\left(-c\sqrt{bq}/\sqrt{a}, -c\sqrt{aq}/\sqrt{b}; q\right)_\infty} \frac{\left(a, b, c^2q^2/a, c^2q^2/b; q^2\right)_\infty}{\left(q, abq, c^2q, c^2q/ab; q^2\right)_\infty}.
\end{aligned} \tag{6.3}
$$

Identity (6.2) is a q-analogue for [3, equation (1)]. It is proved by using the contiguous relation

$$
\begin{aligned}
{}_{r+1}\Phi_r\left[\begin{matrix} x,y,(A); \\ u/x,u/y,(B); \end{matrix} q,z\right] &= {}_{r+1}\Phi_r\left[\begin{matrix} xq,y/q,(A); \\ u/xq,uq/y,(B); \end{matrix} q,z\right] \\
&+ \frac{(1-u)(1-u/xy)(-x+y/q)z}{(1-u/x)(1-u/y)(1-u/xq)(1-uq/y)} \frac{\prod_{i=1}^{r-1}(1-A_i)}{\prod_{i=1}^{r-2}(1-B_i)} {}_{r+2}\Phi_{r+1}\left[\begin{matrix} xq,y,uq,(qA); \\ uq/x,uq^2/y,u,(qB); \end{matrix} q,z\right]
\end{aligned}
\tag{6.4}
$$

with $r=7$, $x=-\sqrt{a\,b}/c\sqrt{q}$, $y=cq$, $u=-c\sqrt{a\,b\,q}$, and (A), (B) such that the left-hand side becomes the ${}_8\Phi_7$ in (6.2). Subsequently, the resulting ${}_8\Phi_7$'s at the right-hand side can be evaluated by (6.1). The contiguous relation (6.4) is equivalent with (5.7).

Likewise, (6.3) is a q-analogue of [3, equation (2)]. It follows from the contiguous relation

$$
\begin{aligned}
{}_{r+1}\Phi_r\left[\begin{matrix} x,(A); \\ y,(B); \end{matrix} q,z\right] &= {}_{r+1}\Phi_r\left[\begin{matrix} xq,(A); \\ y/q,(B); \end{matrix} q,\frac{z}{q}\right] \\
&- \frac{(1-xy)z}{(1-y)(1-y/q)q} \frac{\prod_{i=1}^{r}(1-A_i)}{\prod_{i=1}^{r-1}(1-B_i)} {}_{r+2}\Phi_{r+1}\left[\begin{matrix} xq,xyq,(qA); \\ yq,xy,(qB); \end{matrix} q,\frac{z}{q}\right]
\end{aligned}
\tag{6.5}
$$

when setting $r=7$, $x=-\sqrt{a\,b}/c\sqrt{q}$, $y=c^2 q$, and the remaining parameters such that the left-hand side becomes the ${}_8\Phi_7$ in (6.3). Once again the resulting ${}_8\Phi_7$'s at the right-hand side can be evaluated by (6.1).

Now in (1.1) choose $a_1=-\sqrt{a\,b}\,c/\sqrt{q}$, $a_2=i\,(a\,b)^{1/4}\sqrt{c}\,q^{3/4}$, $a_3=-i\,(a\,b)^{1/4}\sqrt{c}\,q^{3/4}$, $a_4=a$, $a_5=b$, $a_6=c$, $a_7=-c$, $a_8=-\sqrt{a\,b\,q}/c$, $b_1=i\,(a\,b)^{1/4}\sqrt{c}/q^{1/4}$, $b_2=-i\,(a\,b)^{1/4}\sqrt{c}/q^{1/4}$, $b_3=-c\sqrt{b\,q}/\sqrt{a}$, $b_4=-c\sqrt{a\,q}/\sqrt{b}$, $b_5=-\sqrt{a\,b\,q}$, $b_6=\sqrt{a\,b\,q}$, $b_7=c^2$, $z=c\sqrt{q}/\sqrt{a\,b}$. The resulting ${}_8\Phi_7$ at the right-hand side can be summed by (6.1). In the limit $b\to 1$, we get

$$
\begin{aligned}
&{}_9\Phi_8\left[\begin{matrix} i\,a^{1/4}\sqrt{c}\,q^{7/4}, -i\,a^{1/4}\sqrt{c}\,q^{7/4}, q, a\,q, c\,q, \\ i\,a^{1/4}\sqrt{c}\,q^{3/4}, -i\,a^{1/4}\sqrt{c}\,q^{3/4}, -\sqrt{a}\,c\,q^{3/2}, -c\,q^{3/2}/\sqrt{a}, -\sqrt{a}\,q^{3/2}, \end{matrix}\right. \\
&\qquad \left.\begin{matrix} -c\,q, -\sqrt{a}\,q^{3/2}/c, -c\sqrt{a\,q}, q; \\ \sqrt{a}\,q^{3/2}, c^2\,q, q^2; \end{matrix} q, \frac{c\sqrt{q}}{\sqrt{a}}\right] = \frac{(1+c\sqrt{q}/\sqrt{a})\,(1-a\,q)\,(1+c\sqrt{a\,q})}{q\,(1-a)\,(1+\sqrt{a}\,c\,q^{3/2})\,(1+c/\sqrt{a\,q})} \\
&\qquad \times \left(\tilde\psi_q\left(-c\sqrt{q}/\sqrt{a}\right) - \tilde\psi_q\left(-c\sqrt{a\,q}\right) + \frac{\tilde\psi_{q^2}(a\,q) - \tilde\psi_{q^2}(q) + \tilde\psi_{q^2}\left(c^2\,q\right) - \tilde\psi_{q^2}\left(c^2\,q/a\right)}{1+q}\right).
\end{aligned}
\tag{6.6}
$$

This identity is a q-analogue for [3, second identity on p. 272] and for

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix} \frac{7}{4}+\frac{a}{4}+\frac{c}{2}, 1, 1+a, 1+c, \frac{3}{2}+\frac{a}{2}-c, \frac{1}{2}+\frac{a}{2}+c, 1; \\ \frac{3}{4}+\frac{a}{4}+\frac{c}{2}, \frac{3}{2}+\frac{a}{2}+c, \frac{3}{2}-\frac{a}{2}+c, \frac{3}{2}+\frac{a}{2}, 1+2c, 2; \end{matrix} -1\right] \\
&\qquad = \frac{(1+a)(a-2c-1)(1+a+2c)}{a(1+a-2c)(3+a+2c)} \left(\psi\left(\frac{1}{2}-\frac{a}{2}+c\right) - \psi\left(\frac{1}{2}+\frac{a}{2}+c\right)\right. \\
&\qquad \left. + \frac{1}{2}\left(\psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{1}{2}\right) + \psi\left(\frac{1+2c}{2}\right) - \psi\left(\frac{1-a+2c}{2}\right)\right)\right),
\end{aligned}
\tag{6.7}
$$

which is seen by replacing a by q^a, c by $-q^c$, and letting $q\uparrow 1$ in (6.6). A nice limiting case results from (6.6) by letting $c\to 0$,

$$
{}_3\Phi_3\left[\begin{matrix} q,aq,q; \\ -\sqrt{a}q^{3/2}, \sqrt{a}q^{3/2}, q^2; \end{matrix} q,-q^2\right] = \frac{(1-aq)}{q\,(1-a)(1+q)}\left(\tilde\psi_{q^2}(aq) - \tilde\psi_{q^2}(q)\right).
\tag{6.8}
$$

Replacing a by q^a and letting $q\uparrow 1$ gives

$$
{}_3F_2\left[\begin{matrix} 1,1,1+a; \\ \frac{3}{2}+\frac{a}{2}, 2; \end{matrix} \frac{1}{2}\right] = \frac{(1+a)}{2\,a}\left(\psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{1}{2}\right)\right).
\tag{6.9}
$$

Next in (1.1) choose $a_1 = -\sqrt{a\,b}\,c/\sqrt{q}$, $a_2 = i\,(a\,b)^{1/4}\sqrt{c}\,q^{3/4}$, $a_3 = -i\,(a\,b)^{1/4}\sqrt{c}\,q^{3/4}$, $a_4 = a$, $a_5 = b$, $a_6 = -c$, $a_7 = c\,q$, $a_8 = -\sqrt{a\,b}/c\sqrt{q}$, $b_1 = i\,(a\,b)^{1/4}\sqrt{c}/q^{1/4}$, $b_2 = -i\,(a\,b)^{1/4}\sqrt{c}/q^{1/4}$, $b_3 = -c\sqrt{b\,q}/\sqrt{a}$, $b_4 = -c\sqrt{a\,q}/\sqrt{b}$, $b_5 = \sqrt{a\,b\,q}$, $b_6 = -\sqrt{a\,b}/\sqrt{q}$, $b_7 = c^2\,q$, $z = c\sqrt{q}/\sqrt{a\,b}$. The resulting ${}_8\Phi_7$ at the right-hand side can be summed by (6.2). Letting $b \to 1$ leads to a q-analogue of [3, identity at the bottom of p. 271],

$$\begin{aligned}
&{}_9\Phi_8\left[\begin{matrix} i\,a^{1/4}\sqrt{c}\,q^{5/4}, -i\,a^{1/4}\sqrt{c}\,q^{5/4}, c\,q, -\sqrt{a}\,q^{3/2}/c, a\,q, q, -c, -c\sqrt{a}/\sqrt{q}, q; \\ i\,(a\,q)^{1/4}\sqrt{c}, -i\,(a\,q)^{1/4}\sqrt{c}, -\sqrt{a\,q}, c^2, -c\,q^{1/2}/\sqrt{a}, -\sqrt{a}\,c\,q^{1/2}, \sqrt{a}\,q^{3/2}, q^2; \end{matrix}\right. \\
&\qquad \left. q, \frac{c}{\sqrt{a\,q}}\right] = \frac{(1-q)\left(1+c^2/\sqrt{a}\,q^{3/2}\right)(1+c)\left(1+c\sqrt{a}/\sqrt{q}\right)}{(1+c/q)\left(1+c\sqrt{a\,q}\right)\left(1+\sqrt{a\,q}\right)} \frac{(a\,q^2, c^2\,q^2, c^2/a, q^2; q^2)_\infty}{(q, a\,q^3, c^2\,q, c^2/a\,q; q^2)_\infty} \\
&\qquad + \frac{(1-c^2/q)\left(1-\sqrt{a\,q}\right)\left(1+\sqrt{a}/\sqrt{q}\right)\left(1+c\sqrt{a}/\sqrt{q}\right)}{(1-a)(1-c)(1+c/q)\left(1+c\sqrt{a\,q}\right)} \left(\tilde{\psi}_q\left(-\frac{c}{\sqrt{a\,q}}\right) - \tilde{\psi}_q\left(-\frac{c\sqrt{a}}{\sqrt{q}}\right)\right. \\
&\qquad \left. + \frac{\tilde{\psi}_{q^2}(a\,q) - \tilde{\psi}_{q^2}(q) + \tilde{\psi}_{q^2}(c^2/q) - \tilde{\psi}_{q^2}(c^2/a\,q)}{1+q}\right).
\end{aligned} \tag{6.10}$$

Replacing a by q^a, c by $-q^c$, in the limit $q \uparrow 1$ we obtain another ${}_7F_6$-evaluation,

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix} \frac{5}{4}+\frac{a}{4}+\frac{c}{2}, \frac{3}{2}+\frac{a}{2}-c, 1+a, 1, c, -\frac{1}{2}+\frac{a}{2}+c, 1; \\ \frac{1}{4}+\frac{a}{4}+\frac{c}{2}, 2c, \frac{1}{2}-\frac{a}{2}+c, \frac{1}{2}+\frac{a}{2}+c, \frac{3}{2}+\frac{a}{2}, 2; \end{matrix} -1\right] \\
&\qquad = \frac{c(2c+a-1)}{2(c-1)(2c+a+1)} \Gamma\left[\begin{matrix} \frac{1}{2}, \frac{a+3}{2}, \frac{2c+1}{2}, \frac{2c-a-1}{2} \\ \frac{a+2}{2}, \frac{2c+2}{2}, \frac{2c-a}{2} \end{matrix}\right] \\
&\qquad - \frac{(1+a)(2c-1)(2c+a-1)}{2\,a\,(c-1)(2c+a+1)} \left(\psi\left(c+\frac{a}{2}-\frac{1}{2}\right) - \psi\left(c-\frac{a}{2}-\frac{1}{2}\right)\right. \\
&\qquad \left. + \frac{1}{2}\left(\psi\left(\frac{1}{2}\right) - \psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{2c-1}{2}\right) + \psi\left(\frac{2c-a-1}{2}\right)\right)\right).
\end{aligned} \tag{6.11}$$

Also letting $c \to 0$ in (6.10) is of interest. We thus obtain an identity related to (6.8),

$$\begin{aligned}
&{}_3\Phi_3\left[\begin{matrix} a\,q, q, q; \\ -\sqrt{a\,q}, \sqrt{a}\,q^{3/2}, q^2; \end{matrix} q, -q\right] \\
&\qquad = \frac{(1-q)}{\left(1+\sqrt{a\,q}\right)} \frac{\left(a\,q^2, q^2; q^2\right)_\infty}{(q, a\,q^3; q^2)_\infty} + \frac{\left(1-\sqrt{a\,q}\right)\left(1+\sqrt{a}/\sqrt{q}\right)}{(1-a)(1+q)} \left(\tilde{\psi}_{q^2}(a\,q) - \tilde{\psi}_{q^2}(q)\right).
\end{aligned} \tag{6.12}$$

Finally, if in (1.1) we choose $a_1 = -\sqrt{a\,b}\,c/\sqrt{q}$, $a_2 = i\,(a\,b)^{1/4}\sqrt{c}\,q^{3/4}$, $a_3 = -i\,(a\,b)^{1/4}\sqrt{c}\,q^{3/4}$, $a_4 = a$, $a_5 = b$, $a_6 = c$, $a_7 = -c$, $a_8 = -\sqrt{a\,b}/c\sqrt{q}$, $b_1 = i\,(a\,b)^{1/4}\sqrt{c}/q^{1/4}$, $b_2 = -i\,(a\,b)^{1/4}\sqrt{c}/q^{1/4}$, $b_3 = -c\sqrt{b\,q}/\sqrt{a}$, $b_4 = -c\sqrt{a\,q}/\sqrt{b}$, $b_5 = -\sqrt{a\,b\,q}$, $b_6 = \sqrt{a\,b\,q}$, $b_7 = c^2\,q$, $z = c\,q^{3/2}/\sqrt{a\,b}$, by applying (6.3) at the right-hand side we obtain

$$\begin{aligned}
&{}_9\Phi_8\left[\begin{matrix} i(a\,b)^{1/4}\sqrt{c}q^{7/4}, -i(a\,b)^{1/4}\sqrt{c}q^{7/4}, aq, bq, \\ i(a\,b)^{1/4}\sqrt{c}q^{3/4}, -i(a\,b)^{1/4}\sqrt{c}q^{3/4}, -\sqrt{b}cq^{3/2}/\sqrt{a}, -\sqrt{a}cq^{3/2}/\sqrt{b}, \end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix} cq, -cq, -\sqrt{a\,b\,q}/c, -c\sqrt{a\,b\,q}, q; \\ -\sqrt{a\,b}q^{3/2}, \sqrt{a\,b}q^{3/2}, c^2q^2, q^2; \end{matrix} q, \frac{cq^{3/2}}{\sqrt{a\,b}}\right] \\
&= \frac{(1-q)(1-abq)\left(1-c^2q\right)\left(1+c\sqrt{a\,q}/\sqrt{b}\right)\left(1+c\sqrt{bq}/\sqrt{a}\right)}{q\,(1-a)(1-b)(1-c)(1+c)\left(1+c\sqrt{q}/\sqrt{a\,b}\right)\left(1+\sqrt{a\,b}\,c\,q^{3/2}\right)} \\
&\quad \times \left(-\frac{c\sqrt{q}}{\sqrt{a\,b}} \frac{\left(-c\sqrt{a\,b\,q}, -c\sqrt{q}/\sqrt{a\,b}; q\right)_\infty}{\left(-c\sqrt{bq}/\sqrt{a}, -c\sqrt{a\,q}/\sqrt{b}; q\right)_\infty} \frac{\left(a, b, c^2q^2/a, c^2q^2/b; q^2\right)_\infty}{\left(q, abq, c^2q, c^2q/ab; q^2\right)_\infty}\right. \\
&\quad \left. + \frac{\left(-c\sqrt{a\,b\,q}, -c\sqrt{q}/\sqrt{a\,b}; q\right)_\infty}{\left(-c\sqrt{bq}/\sqrt{a}, -c\sqrt{a\,q}/\sqrt{b}; q\right)_\infty} \frac{\left(aq, bq, c^2q/a, c^2q/b; q^2\right)_\infty}{\left(q, abq, c^2q, c^2q/ab; q^2\right)_\infty} - 1\right).
\end{aligned} \tag{6.13}$$

Letting $b \to 1$ yields a q-analogue of [3, identity at the top of p. 272]

$$
{}_9\Phi_8\left[\begin{matrix} ia^{1/4}\sqrt{c}q^{7/4}, -ia^{1/4}\sqrt{c}q^{7/4}, aq, q, cq, -cq, -\sqrt{a\,q}/c, \\ ia^{1/4}\sqrt{c}q^{3/4}, -ia^{1/4}\sqrt{c}q^{3/4}, -cq^{3/2}/\sqrt{a}, -\sqrt{a}cq^{3/2}, -\sqrt{a}q^{3/2}, \sqrt{a}q^{3/2}, c^2q^2, \end{matrix}\right.
$$

$$
\left.\begin{matrix} -c\sqrt{a\,q}, q; \\ q^2; \end{matrix}\; q, \frac{cq^{3/2}}{\sqrt{a}}\right] = -\frac{c(1-q)\left(1+c\sqrt{a\,q}\right)}{\sqrt{a\,q}(1-c)(1+c)\left(1+\sqrt{a}cq^{3/2}\right)}\frac{\left(q^2, aq^2, c^2q^2, c^2q^2/a; q^2\right)_\infty}{\left(c^2q/a, q, aq^3, c^2q^3; q^2\right)_\infty}
$$

$$
+\frac{(1-aq)\left(1-c^2q\right)\left(1+c\sqrt{a\,q}\right)}{q\,(1-a)(1-c)(1+c)\left(1+\sqrt{a}cq^{3/2}\right)}\left(\tilde{\psi}_q\left(-\frac{c\sqrt{q}}{\sqrt{a}}\right) - \tilde{\psi}_q\left(-c\sqrt{a\,q}\right)\right.
$$

$$
\left. + \frac{\tilde{\psi}_{q^2}(aq) - \tilde{\psi}_{q^2}(q) + \tilde{\psi}_{q^2}\left(c^2q\right) - \tilde{\psi}_{q^2}\left(c^2q/a\right)}{1+q}\right). \tag{6.14}
$$

A ${}_7F_6$-evaluation related to (6.7) and (6.11) is obtained from (6.14) by replacing a by q^a, c by $-q^c$, and then letting $q \uparrow 1$,

$$
\begin{aligned}
{}_7F_6&\left[\begin{matrix} \frac{7}{4}+\frac{a}{4}+\frac{c}{2}, 1+a, 1, 1+c, \frac{1}{2}+\frac{a}{2}-c, \frac{1}{2}+\frac{a}{2}+c, 1; \\ \frac{3}{4}+\frac{a}{4}+\frac{c}{2}, \frac{3}{2}-\frac{a}{2}+c, \frac{3}{2}+\frac{a}{2}+c, \frac{3}{2}+\frac{a}{2}, 2+2c, 2; \end{matrix} -1\right] \\
&= \frac{(2c+a+1)}{2\,c\,(2c+a+3)}\Gamma\left[\begin{matrix} \frac{1}{2}, \frac{3+a}{2}, \frac{3+2c}{2}, \frac{1-a+2c}{2} \\ \frac{2+a}{2}, \frac{2+2c}{2}, \frac{2-a+2c}{2} \end{matrix}\right] \\
&\quad - \frac{(1+a)(1+2c)(2c+a+1)}{2ac(2c+a+3)}\left(\psi\left(\frac{1}{2}+\frac{a}{2}+c\right) - \psi\left(\frac{1}{2}-\frac{a}{2}+c\right)\right. \\
&\quad \left. + \frac{1}{2}\left(\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{1+2c}{2}\right) + \psi\left(\frac{1-a+2c}{2}\right)\right)\right).
\end{aligned} \tag{6.15}
$$

On the other hand, letting $c \to 1$ in (6.13) leads to a q-analogue of [3, equation (7)]

$$
{}_9\Phi_8\left[\begin{matrix} i(a\,b)^{1/4}q^{7/4}, -i(a\,b)^{1/4}q^{7/4}, aq, bq, q, \\ i(a\,b)^{1/4}q^{3/4}, -i(a\,b)^{1/4}q^{3/4}, -\sqrt{b}q^{3/2}/\sqrt{a}, -\sqrt{a}q^{3/2}/\sqrt{b}, -\sqrt{a\,b}q^{3/2}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} -q, -\sqrt{a\,b\,q}, -\sqrt{a\,b\,q}, q; \\ \sqrt{a\,b}q^{3/2}, q^2, q^2; \end{matrix}\; q, \frac{q^{3/2}}{\sqrt{a\,b}}\right] = \frac{(1-q)(1-abq)\left(1+\sqrt{a\,q}/\sqrt{b}\right)\left(1+\sqrt{b\,q}/\sqrt{a}\right)}{2(1-a)(1-b)\left(1+\sqrt{q}/\sqrt{a\,b}\right)q\left(1+\sqrt{a\,b}q^{3/2}\right)}
$$

$$
\times\left(\frac{(1-q)\sqrt{q}}{\sqrt{a\,b}}\frac{\left(-\sqrt{a\,b\,q}, -\sqrt{q}/\sqrt{a\,b}; q\right)_\infty}{\left(-\sqrt{b\,q}/\sqrt{a}, -\sqrt{a\,q}/\sqrt{b}; q\right)_\infty}\frac{\left(a, b, q^2/a, q^2/b; q^2\right)_\infty}{\left(q, abq, q, q/ab; q^2\right)_\infty}\right.
$$

$$
\times\tilde{\psi}q\left(-\frac{\sqrt{a\,q}}{\sqrt{b}}\right) - \tilde{\psi}_q\left(-\frac{\sqrt{q}}{\sqrt{a\,b}}\right) + \tilde{\psi}q\left(-\frac{\sqrt{b\,q}}{\sqrt{a}}\right) - \tilde{\psi}q\left(-\sqrt{a\,b\,q}\right) + \frac{2}{1+q}\left(\tilde{\psi}q^2(q) + \tilde{\psi}q^2\left(\frac{q}{ab}\right)\right)
$$

$$
-\frac{2}{1+q}\frac{\left(-\sqrt{a\,b\,q}, -\sqrt{q}/\sqrt{a\,b}; q\right)_\infty}{\left(-\sqrt{b\,q}/\sqrt{a}, -\sqrt{a\,q}/\sqrt{b}; q\right)_\infty}\left(\frac{\left(aq, bq, q/a, q/b; q^2\right)_\infty}{\left(q, abq, q, q/ab; q^2\right)_\infty}\left(\tilde{\psi}q^2\left(\frac{q}{a}\right) + \tilde{\psi}q^2\left(\frac{q}{b}\right)\right)\right.
$$

$$
\left.\left.-\frac{\sqrt{q}}{\sqrt{a\,b}}\frac{\left(a, b, q^2/a, q^2/b; q^2\right)_\infty}{\left(q, abq, q, q/ab; q^2\right)_\infty}\left(\tilde{\psi}q^2\left(\frac{q^2}{a}\right) + \tilde{\psi}q^2\left(\frac{q^2}{b}\right)\right)\right)\right). \tag{6.16}
$$

Again, the series in (6.6), (6.7), (6.10)–(6.16) are very well-poised series with cancelling leading parameter.

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